

Kinetic Equation for a Weakly Interacting Classical Electron Gas

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A kinetic equation is derived for the two-time phase space correlation function in a dilute classical electron gas in equilibrium. The derivation is based on a density expansion of the correlation function and the resummation of the most divergent terms in each order in the density. It is formally analogous to the ring summation used in the kinetic theory of neutral fluids. The kinetic equation obtained is consistent to first order in the plasma parameter and is the generalization of the linearized Balescu–Guernsey–Lenard operator to describe spatially inhomogeneous equilibrium fluctuations. The importance of consistently treating static correlations when deriving a kinetic equation for an electron gas is stressed. A systematic derivation as described here is needed for a further generalization to a kinetic equation that includes mode-coupling effects. This will be presented in a future paper.

KEY WORDS: Electron gas; Balescu–Guernsey–Lenard kinetic equation; time correlation functions.

1. INTRODUCTION

Cluster expansion techniques have been a very powerful tool for studying equilibrium time correlation functions in neutral gases at low and moderate densities.⁽¹⁾ Most of the developments in this field are, however, restricted to gases with a short-range intermolecular potential. In this and a following paper⁽²⁾ the same methods are employed to study equilibrium time

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correlation functions in a gas of charged particles interacting through a weak long-range potential. Specifically, the case of Coulomb interaction will be considered. The strength of the coupling in a Coulomb gas is measured in terms of the plasma parameter, ε_p , representing the ratio of potential to kinetic energy, or equivalently, the number of particles in the Debye sphere, i.e., $\varepsilon_p = (4\pi n \lambda_D^3)^{-1}$. Here n is the number density, e the electron charge, and $\lambda_D = (4\pi n e^2 \beta)^{-1/2}$, with $T = (k_B \beta)^{-1}$ the temperature, is the Debye shielding length, which measures the effective range of the interaction. For small values of the plasma parameter the decay of equilibrium fluctuations in an electron gas is described by the linearized Balescu–Guernsey–Lenard (BGL) kinetic equation, which takes into account effective two-body collisions and is in this sense analogous to the Boltzmann equation for neutral gases. The spatially homogeneous BGL equation was first obtained in its nonlinear form by Balescu⁽³⁾ (and, independently, by Lenard and also by Guernsey) using a nonequilibrium cluster expansion method.

The linear BGL equation can be generalized in two directions. First, it is of interest to obtain an inhomogeneous form of the BGL equation to describe spatially varying equilibrium fluctuations. Secondly, in order to describe moderately dense, instead of dilute, electron gases one needs to derive higher-order plasma parameter corrections to the BGL equation and take into account additional collective effects not contained in the BGL operator itself.

In this paper a systematic derivation of the linearized BGL kinetic equation for spatially inhomogeneous equilibrium fluctuations is presented. The derivation is based on a detailed analysis of the cluster expansion of the exact N -particle collision operator (in the spirit of Balescu's first derivation) and employs the techniques developed in the theory of neutral fluids.⁽¹⁾ The approximations introduced are well controlled and the method can be extended to yield corrections to the linear BGL equation that include hydrodynamic mode-coupling effects. This will be done in a subsequent publication.⁽²⁾

The need of a careful derivation of the linearized BGL equation as a starting point for its generalization is one of the motivations of the present work. The calculation has, however, also intrinsic interest, since it illustrates the importance of consistently treating static correlations when deriving a kinetic equation for an inhomogeneous electron gas.

The problem of a systematic derivation of the linearized BGL equation for equilibrium time correlation functions using cluster expansion methods has been considered before by Bartis and Oppenheim⁽⁴⁾ for the spatially homogeneous case. They considered a gas with an intermolecular potential consisting of a weak long-range part and a strong short-range part, and

neglected all terms explicitly containing equilibrium correlations. For the spatially homogeneous case considered there these terms do not contribute to the leading part of the density expansion of the transport coefficients, whose evaluation was the objective of the paper. The procedure of Bartis and Oppenheim is, however, not suitable for constructing a generalized BGL equation describing spatially inhomogeneous equilibrium correlations nor is it suitable for deriving corrections to the BGL operator⁽²⁾ itself even for the homogeneous case. In both cases a systematic analysis of static correlations as well as of dynamical events is needed. This will be discussed in detail below and elsewhere.⁽²⁾

A generalized spatially inhomogeneous BGL equation has been obtained before by truncation of the BBGKY hierarchy.^(5,6) The hierarchy method gives the correct form of the generalized BGL operator. In addition, it generates "singular" corrections to the Vlasov operator (i.e., corrections that are as divergent as the Vlasov operator itself at large interparticle separation) and terms that have the form of initial condition corrections. In Ref. 6 both such correction terms were neglected on the grounds of their ordering in the plasma parameter. By discussing in detail the connection between the hierarchy and the cluster expansion method, where such terms do not appear, we are able to show that they are consistently canceled in the hierarchy formulation as well.

Finally, formal kinetic equations for Coulomb fluids have been obtained by employing the so-called algebraic methods,⁽⁷⁻⁹⁾ originated by Lebowitz, Percus, and Sykes for neutral gases.⁽¹⁰⁾ These approaches lead to an exact but formal kinetic equation for the time correlation function of interest, containing a formal expression for the collision operator. By assuming an analytic expansion in powers of the plasma parameter the collision kernel can, however, be evaluated explicitly to the lowest order. The resulting kinetic equation is, for long-time and large distances, identical to that obtained here [see Eqs. (3.13a) and (3.15)].

The algebraic methods have been implemented in the literature by introducing simple approximations to the collision kernel (disconnected approximation, effective interaction approximation).⁽⁷⁻⁹⁾ These have yielded predictions for the dynamical properties of dense Coulomb liquids that compare well with the results of computer experiments.⁴ When such approximate collision operators are evaluated to leading order in the plasma parameter, the result obtained here is again recovered. In this sense, our calculation provides an independent check for such methods. The approximations mentioned above fail, however, to predict correctly a certain type of plasma parameter corrections to the BGL operator, that is,

⁴ For a review see Ref. 11.

those containing mode-coupling effects, that are responsible for the asymptotic slow long-time decay of the correlation functions. Our method, on the contrary, is based on a systematic expansion in a small parameter and appears well suited to analyze such corrections to the BGL operator and obtain a consistent theory of mode-coupling effects in an electron gas. This will be discussed in a following paper.⁽²⁾

The technique used to derive the BGL equation in this paper is motivated by the structure of the cluster expansion in neutral fluids. There it is known that such a naive expansion is not well defined: each term of the expansion beyond the first two terms contains a contribution that diverges in the long-time limit. The leading divergences are due to a special class of dynamical events containing certain sequences of s binary collisions among s particles, known as ring collisions. Owing to the expansion in the density, such collisions are allowed to take place over arbitrarily long times and the collective effect of a mean free path damping of the particle trajectories in these sequences due to the other particles is not taken into account. To include this effect and obtain a well-behaved collision operator it is necessary to carry out a resummation of all the most divergent terms in each order in the density.⁽¹⁾

It is expected that similar difficulties will appear in systems with weak long-range interactions.⁽¹²⁾ Furthermore, in this case, owing to the long range of the potential, additional divergences are encountered already in the density expansion of the equilibrium properties. In equilibrium a well-defined expansion is obtained by resumming the most divergent terms in each order in the density, as was first shown by Mayer.⁽¹³⁾ In first approximation this leads to the same results as the phenomenological Debye-Hückel theory. Again, the divergences in the equilibrium density expansion indicate that a collective effect, here the screening of the Coulomb interaction at large distances, is not properly taken into account by the density expansion.

In the nonequilibrium cluster expansion for systems with long-range forces, divergences of both a dynamical (as in neutral fluids) and a static (as in equilibrium) origin appear. A resummation is needed to remove both types of singular behavior. In this paper the case of a weak long-ranged potential is considered and only divergences associated with small angle scattering at large interparticle separation are discussed. The effects of the strong short-range part of the interaction potential are not considered here.^(4,14)

The central quantity of interest here is the two-time time correlation function of phase space fluctuations, $C(1t; 1')$, from which all the relevant two-particle time correlation functions may be obtained. The objective is to derive a closed kinetic equation for $C(1t; 1')$, i.e., identify a well-behaved

collision operator that governs the long time evolution of $C(1t; 1')$ for small ε_p . In Section 2 the quantities of interest are defined and the cluster expansion of $\partial C(1t; 1')/\partial t$ is outlined. The rearrangements needed to eliminate the divergences in this “naive density expansion” and the resulting new cluster expansion for $C(1t; 1')$ are also discussed there. In Section 3 the first few terms in this new cluster expansion are analyzed and a criterion to identify the class of most divergent collision sequences in each order in the density is given. A resummation of these most divergent contributions is carried out. As a result, a generalized BGL operator is identified and discussed. In Section 4 the connection with the BBGKY hierarchy method is discussed. A consistent truncation of the hierarchy is performed, that leads to an equation that is identical to that obtained from the cluster expansion. Finally, the results obtained in this paper, as well as some open questions, are discussed in Section 5.

2. THE CLUSTER EXPANSION

The system considered is the classical electron gas, i.e., a gas of N electrons of charge e and mass m , immersed in a uniform neutralizing background. The gas is enclosed in a volume Ω and is in equilibrium at temperature T .

All the two-particle time correlation functions of interest may be expressed in terms of the time correlation function of phase space fluctuations, defined as

$$\begin{aligned} C(1t; 1') &= \langle \delta\psi(1, t) \delta\psi(1', 0) \rangle \\ &= \lim_{\substack{N, \Omega \rightarrow \infty \\ N/\Omega = n}} \int dx^N \delta\psi(1, t) \delta\psi(1', 0) \rho(x^N) \end{aligned} \quad (2.1)$$

where $\delta\psi(1, t)$ represents the fluctuation in the single particle phase space density, $\psi(1, t)$,

$$\delta\psi(1, t) = \psi(1, t) - \langle \psi(1, t) \rangle \quad (2.2)$$

and $\psi(1, t)$ is the number of particles at the field point $1 = x_1 = (\mathbf{r}_1, \mathbf{v}_1)$ in the six-dimensional single particle phase space at time t , given by

$$\psi(1, t) = \sum_{i=1}^N \delta(1 - X_i(t)) \quad (2.3)$$

Here $X^N = X_1, \dots, X_N$ denotes the phase of the N electrons 1, 2, ..., N , with $X_i = (\mathbf{R}_i, \mathbf{V}_i)$, and $X_i(t)$ is the phase of the i th particle at time t as evolved

from its initial value, $X_i = X_i(0)$. Only the dependence on the field variable is indicated explicitly in the function $\psi(1, t)$. The angular brackets indicate an average over the canonical probability density, $\rho(X^N)$. The average value of $\psi(1, t)$ in Eq. (2.2) is the single-particle distribution function characterizing the average number of particles at the point 1,

$$f_1(1) = \langle \psi(1, t) \rangle = \langle \psi(1, 0) \rangle \quad (2.4)$$

For a system in equilibrium, in the absence of external fields, $f_1(1)$ is independent of position and time and is simply given by

$$f_1(1) = n\phi(v_1) \quad (2.5)$$

where $\phi(v) = (m\beta/2\pi)^{3/2} \exp(-m\beta v^2/2)$ is the Maxwell velocity distribution function, and $n = N/\Omega$ is the number density. The time evolution of any phase function $A(X^N)$ is governed by the N -particle streaming operator $S_t(X^N)$, according to

$$A(X^N(t)) = S_t(X^N) A(X^N) \quad (2.6)$$

The streaming operator may be formally written as

$$S_t(X^N) = e^{tL(X^N)} \quad (2.7)$$

where $L(X^N)$ is the N -particle Liouville operator, defined as

$$L(X^N) = \{ \quad, H(X^N) \} \quad (2.8)$$

Here the curly brackets denote the Poisson brackets with the Hamiltonian of the system, $H(X^N)$,

$$H(X^N) = \sum_{i=1}^N \frac{1}{2} m V_i^2 + \sum_{i < j=1}^N V(R_{ij}) \quad (2.9)$$

where $R_{ij} = |\mathbf{R}_i - \mathbf{R}_j|$. The interaction among the electrons is the Coulomb interaction, $V(R) = e^2/R$. Most of the formal manipulations hold, however, for a general central long-range pair potential. For systems with a continuous and differentiable intermolecular potential the Liouville operator is given by

$$L(X^N) = \sum_{i=1}^N L_0(X_i) - \sum_{i < j=1}^N \theta_{ij} \quad (2.10)$$

with

$$L_0(X_i) = \mathbf{V}_i \cdot \frac{\partial}{\partial \mathbf{R}_i} \quad (2.11a)$$

$$\theta_{ij} = \theta(X_i, X_j) = \frac{1}{m} \frac{\partial V(R_{ij})}{\partial \mathbf{R}_i} \cdot \left(\frac{\partial}{\partial \mathbf{V}_i} - \frac{\partial}{\partial \mathbf{V}_j} \right) \quad (2.11b)$$

Making use of Eq. (2.6) and of the invariance of equilibrium averages under time translation, it is convenient to rewrite Eq. (2.1) more explicitly as follows:

$$C(1t; 1') = \int dX^N \delta\psi(1, 0) S_{-t}(X^N) \rho(X^N) \delta\psi(1', 0) \quad (2.12)$$

where here and in the following the thermodynamic limit of the integral is intended.

The objective now is to obtain a kinetic equation for $C(1t; 1')$. In particular, we are interested in determining the long time ($t \gg \tau_c$, with τ_c a characteristic collision time⁵) and large distance ($|\mathbf{r}_1 - \mathbf{r}'_1| \gg \lambda_D$) behavior of the two-point correlation function, $C(1t; 1')$. It is convenient to use as the starting point an expression for $\partial C(1t; 1')/\partial t$ that is easily obtained by differentiating Eq. (2.12) with respect to time,

$$\begin{aligned} \frac{\partial}{\partial t} C(1t; 1') &= - \int dX^N N \delta(1 - X_1) L(X^N) S_{-t}(X^N) \rho(X^N) \delta\psi(1', 0) \\ &= -L_0(1) C(1t; 1') + N(N-1) \\ &\quad \times \int dX^{N-1} \theta(1X_2) S_{-t}(1X^{N-1}) \rho(1X^{N-1}) \delta\psi(1', 0) \end{aligned} \quad (2.13a)$$

to be solved with the initial condition

$$C(1, t=0; 1') = C(1; 1') \quad (2.13b)$$

where $C(1; 1')$ is the equal time correlation function [cf. Eq. (2.12)]. The time evolution of $C(1t; 1')$ may be studied by performing a cluster expansion of the second term on the right-hand side of Eq. (2.13), following closely the procedure employed by Dorfman and Cohen for neutral fluids.⁽¹⁾ In the rest of this section the cluster expansion of Eq. (2.13) is performed. This is done in two steps: (1) by expanding the N -particle streaming operator in terms of Ursell cluster functions; (2) by performing a density expansion of the equilibrium correlation functions. The resulting expansion for $\partial C(1t; 1')/\partial t$, known as the "naive density expansion," tends to diverge term by term in the long-time limit. This divergence, however, is not the crucial one and will be shown to be easily removed by a suitable resummation.

⁵ The concept of collision time is well defined even for a Coulomb gas, where τ_c can be *a posteriori* identified with the inverse of the characteristic collision frequency associated with the BGL operator.

The cluster expansion of the N -particle streaming operator is

$$S_{-t}(X^N) = U_t(X_1, X_2) S_{-t}(X^{N-2}) + \sum_{i=3}^N U_t(X_1, X_2 | X_i) S_{-t}(X^{N-3}) \\ + \sum_{3 \leq i < j \leq N} U_t(X_1, X_2 | X_i, X_j) S_{-t}(X^{N-4}) + \dots \quad (2.14)$$

The explicit form of the operators $U_t(X_1, X_2 | X_3, \dots, X_s)$ is easily obtained by inverting the set of equations resulting from letting $N=2, 3, \dots$ in Eq. (2.14). They are given by

$$U_t(X_1, X_2) = S_{-t}(X_1, X_2) \\ U_t(X_1, X_2 | X_3) = S_{-t}(X_1, X_2, X_3) - S_{-t}(X_1, X_2) S_{-t}(X_3) \quad (2.15)$$

etc. Inserting the cluster expansion (2.14) into Eq. (2.13) and making use of Liouville's theorem, the following expansion for $\partial C(1t; 1')/\partial t$ is obtained:

$$\frac{\partial}{\partial t} C(1t; 1') = -L_0(1) C(1t; 1') \\ + \sum_{s=2}^{\infty} \frac{1}{(s-2)!} \int dX_2 \cdots dX_s \theta(1, X_2) \\ \times U_t(1, X_2 | X_3, \dots, X_s) I_s(1, X_2, \dots, X_s; 1') \quad (2.16)$$

In Eq. (2.16) all the particle variables X_i , for $i=2, 3, \dots, s$, are integrated over and no longer need to be distinguished from the field variables. In the following, when no confusion can arise, we will use for convenience the same notation for the particle variables as for the field variables.

The functions I_s can be expressed in terms of the equilibrium distribution functions of the electron gas. The first few I_s are given by

$$I_2(12; 1') = (1 + P_{12}) f_1(2) C(1; 1') + C(12; 1') \\ I_3(123; 1') = (1 + P_{12} + P_{13}) f_2(23) C(1; 1') + (1 + P_{13} + P_{23}) f_1(3) C(12; 1') \\ + C(123; 1') \quad (2.17)$$

etc., where P_{ij} is a permutation operator that interchanges the labels of particles i and j , and the $C(1, \dots, s; 1')$ are the equal time correlation functions of the phase space densities defined by

$$C(1, \dots, s; 1') = \langle \delta\psi(1, \dots, s) \delta\psi(1') \rangle \quad (2.18)$$

where

$$\delta\psi(1, \dots, s) = \sum_{1 \leq i_1 < \dots < i_s \leq N} \delta(1 - X_{i_1}) \cdots \delta(s - X_{i_s}) \quad (2.19)$$

In Eqs. (2.17) the $f_s(1, \dots, s)$ are the equilibrium reduced distribution functions, defined as

$$\begin{aligned} f_s(X_1, \dots, X_s) &= \frac{N!}{(N-s)!} \int dX_{s+1} \cdots dX_N \rho(X^N) \\ &= n^s \phi(V_1) \cdots \phi(V_s) g_s(\mathbf{R}_1, \dots, \mathbf{R}_s) \end{aligned} \quad (2.20)$$

where g_s is the configurational part of the distribution function. The corresponding cluster functions $G_s(1, \dots, s)$ are also needed in the following. The latter are defined in terms of the f_s in the usual way, i.e.,

$$f_1(1) = f_1(1) \quad (2.21a)$$

$$f_2(12) = f_1(1) f_1(2) + G_2(12) \quad (2.21b)$$

$$\begin{aligned} f_3(123) &= f_1(1) f_1(2) f_1(3) + (1 + P_{12} + P_{13}) f_1(1) G_2(23) \\ &\quad + G_3(123) \end{aligned} \quad (2.21c)$$

etc., where

$$G_s(X_1, \dots, X_s) = n^s \phi(V_1) \cdots \phi(V_s) h_s(\mathbf{R}_1, \dots, \mathbf{R}_s) \quad (2.22)$$

and h_s is the configurational part of the cluster function. The equal time correlation functions $C(1, \dots, s; 1')$ can be expressed in terms of the equilibrium distribution functions as

$$C(1; 1') = \delta(1 - 1') f_1(1) + G_2(1, 1') \quad (2.23a)$$

$$C(12; 1') = (1 + P_{12}) \delta(1 - 1') G_2(1, 1') + G_3(1, 2, 1') \quad (2.23b)$$

etc.

In order to obtain an expansion of the right-hand side of Eq. (2.16) in powers of the density n , it is now necessary to perform a density expansion of the equilibrium distribution functions contained in the I_s . Before doing this it is, however, convenient to put Eq. (2.16) in a slightly different form. As previously stated, the objective here is to derive a kinetic equation for $C(1t; 1')$ and identify the kinetic operator that governs its long-time evolution from the initial condition $C(1; 1')$. It is then useful to introduce the initial condition $C(1; 1')$ explicitly on the right-hand side of Eq. (2.16) by defining an inverse operator $C^{-1}(1; 1')$ as

$$\int d1'' C^{-1}(1; 1'') C(1''; 1') = \delta(1 - 1') \quad (2.24)$$

The solution for $C^{-1}(1; 1')$ is

$$C^{-1}(1; 1') = \frac{1}{f_1(1)} \delta((1 - 1') - c(|\mathbf{r}_1 - \mathbf{r}'_1|)) \quad (2.25)$$

where $c(r)$ is the direct correlation function, defined through the Ornstein-Zernike equation,

$$c(r_{12}) = h_2(r_{12}) - n \int d\mathbf{r}_3 h_2(r_{13}) c(r_{32}) \quad (2.26)$$

and $h_2(r)$ is the pair correlation function, defined by Eqs. (2.21b) and (2.22) for $s=2$. Inserting Eq. (2.24) into Eq. (2.16), the latter may be rewritten

$$\begin{aligned} \frac{\partial}{\partial t} C(1t; 1') &= -L_0(1) C(1t; 1') \\ &+ \sum_{s=2}^{\infty} \frac{1}{(s-2)!} \int d2 \cdots ds \theta_{12} \\ &\times U_t(1, 2 | 3, \dots, s) W_s(1, \dots, s) C(1; 1') \end{aligned} \quad (2.27)$$

where

$$W_s(1, \dots, s) = \int d1'' \int d1''' I_s(1, \dots, s; 1'') C^{-1}(1''; 1''') P_{1,1'''} \quad (2.28)$$

acts as an integral operator on functions of 1. The equilibrium distribution functions appearing in W_s may now be expanded in powers of the density by using the well-known density expansion of the equilibrium distribution functions. The density expansion of the operators W_s is obtained in Appendix A, with the result

$$W_s(1, \dots, s) = n^{s-1} \sum_{l=0}^{\infty} n^l W_s^{(l)}(1, \dots, s) \quad (2.29)$$

with

$$W_s^{(l)}(1, \dots, s) = \sum_{i=1}^s P_{1i} \phi(v_2) \cdots \phi(v_s) g_s^{(0)}(\mathbf{r}_1, \dots, \mathbf{r}_s) \quad (2.30)$$

Here $g_s^{(0)}$ is the configurational part of the s -particle distribution function to lowest order in the density, given in Eq. (A3); for $l \geq 1$ the coefficients $W_s^{(l)}$ have the form

$$W_s^{(l)}(1, \dots, s) = \int d(s+1) \cdots d(s+l) W_s^{(l)}(1, \dots, s | s+1, \dots, s+l) \quad (2.31)$$

and are given in Appendix A for $l=2, 3$. When Eq. (2.29) is inserted into Eq. (2.27) a formal density expansion of $\partial C(1t; 1')/\partial t$ is obtained. Because of the occurrence of time convolutions, it is more convenient for the following discussion to analyze the Laplace transform of $C(1t; 1')$, defined

$$C(1z; 1') = \int_0^\infty dt e^{-zt} C(1t; 1') \quad (2.32)$$

for $\text{Re}z > 0$. In particular, we will be interested in the behavior of $C(1z; 1')$ as $z \rightarrow 0^+$. Equation (2.27) may then be written

$$[z + L_0(1)] C(1z; 1') = \left[1 + \sum_{s=2}^{\infty} n^{s-1} \mathcal{A}_s(1z) \right] C(1; 1') \quad (2.33)$$

where

$$\mathcal{A}_2(1z) = \int d2\theta_{12} U_z(12) W_2^{(0)}(12) \quad (2.34a)$$

$$\mathcal{A}_3(1z) = \int d2d3\theta_{12} \{ U_z(12 | 3) W_3^{(0)}(123) + U_z(12) W_2^{(1)}(12 | 3) \} \quad (2.34b)$$

$$\begin{aligned} \mathcal{A}_4(1z) = & \frac{1}{2!} \int d2d3d4\theta_{12} [U_z(12 | 34) W_4^{(0)}(1234) \\ & + 2U_z(12 | 3) W_3^{(1)}(123 | 4) + 2U_z(12) W_2^{(2)}(12 | 34)] \end{aligned} \quad (2.34c)$$

etc. Here $U_z(12 | 3 \cdots s)$ is the Laplace transform of $U_i(12 | 3 \cdots s)$. Equation (2.33) has the desired form of a cluster expansion of the kinetic operator governing the z dependence of $C(1z; 1')$. It cannot, however, be used to determine the small- z behavior of $C(1z; 1')$ since it contains divergences as $z \rightarrow 0^+$. An analysis of the right-hand side of Eq. (2.33) reveals that each \mathcal{A}_s diverges as $z \rightarrow 0^+$ and that the most divergent contributions to \mathcal{A}_s come from sequences of $s-1$ uncorrelated binary collisions among s particles, leading to a $z^{-(s-1)}$ divergence. Here, as elsewhere in this paper, z^0 stands for a behavior $\sim \ln z$. These divergences are precisely the same that appear in the cluster expansion of the collision operator for neutral fluids.⁽¹⁾ As in that case they can be eliminated by using an inversion procedure due to Zwanzig.⁽¹⁵⁾ First, Eq. (2.33) is rewritten

$$\left[G_0(1, z) + \sum_{s=2}^{\infty} n^{s-1} G_0(1, z) \mathcal{A}_s(1z) \right]^{-1} C(1z; 1') = C(1; 1') \quad (2.35)$$

where $G_0(1, z)$ is the free particle propagator,

$$G_0(1, z) = [z + L_0(1)]^{-1} \quad (2.36)$$

Then, a new set of collision operators B_s is defined by introducing the density expansion of the inverse operator on the left-hand side of Eq. (2.35), i.e.,

$$\left[G_0(1, z) + \sum_{s=2}^{\infty} n^{s-1} \mathcal{A}_s(1z) \right]^{-1} = \left[G_0^{-1}(1, z) - \sum_{s=2}^{\infty} n^{s-1} B_s(1z) \right] \quad (2.37)$$

This leads to the following kinetic equation for $C(1z; 1')$:

$$\left[z + L_0(1) - \sum_{s=2}^{\infty} n^{s-1} B_s(1z) \right] C(1z; 1') = C(1; 1') \quad (2.38)$$

The operators B_s are easily obtained by equating terms of equal order in the density on the two sides of Eq. (2.37), with the result

$$B_2 = \mathcal{A}_2 G_0^{-1}(1, z) \quad (2.39a)$$

$$B_3 = \{ \mathcal{A}_3 - [\mathcal{A}_2]^2 \} G_0^{-1}(1, z) \quad (2.39b)$$

$$B_4 = \{ \mathcal{A}_4 - \mathcal{A}_3 \mathcal{A}_2 - \mathcal{A}_2 \mathcal{A}_3 + [\mathcal{A}_2]^2 \} G_0^{-1}(1, z) \quad (2.39c)$$

etc. The operators B_s may be written in a more convenient form by making use of Liouville's theorem and of operator identities given in Ref. 1, with the result

$$B_2(1, z) = \int d2 \theta_{12} G(12, z) W_2^{(0)}(12) G_0^{-1}(1, z) \quad (2.40a)$$

$$\begin{aligned} B_3(1, z) = & \int d2 d3 \theta_{12} G(12, z) [(1 + P_{12}) \theta_{13} G(123, z) W_3^{(0)}(123) \\ & - W_2^{(0)}(12) \theta_{13} G(13, z) W_2^{(0)}(13) + W_2^{(1)}(12 | 3)] G_0^{-1}(1, z) \end{aligned} \quad (2.40b)$$

etc., where $G(1 \cdots s, z)$ is an s -particle propagator,

$$G(1 \cdots s, z) = [z + L(1 \cdots s)]^{-1} \quad (2.41)$$

The collision sequences that lead to the leading small z divergence of \mathcal{A}_s do not appear in the B_s . In the diagrammatic language introduced in the next section such dynamical events correspond to singly connected diagrams that are canceled in B_s , where only irreducible diagrams appear. It will, however, be shown in the next section that the B_s still contain small z divergences. A resummation of such divergences is needed to identify a well-defined collision operator for $z \rightarrow 0^+$. This then leads to the generalized BGL kinetic operator.

3. DERIVATION OF THE INHOMOGENEOUS BGL EQUATION VIA CLUSTER EXPANSION

As discussed in the Introduction, two kind of divergences are expected to appear in the analysis of the collision operators B_s for a system with long-range forces: divergences of static origin associated with the long range of the potential, as already present in the equilibrium density expansion, and divergences associated with a class of dynamical processes, analogous to those appearing in neutral fluids. An appropriate resummation will be necessary to remove both these divergences.

Here only the case of a *weak* long-range potential is considered and only dynamical divergences associated with small angle scattering processes at large interparticle separation will be discussed.

In this section the first few collision operators B_s are examined in some detail. To carry out this analysis, the right-hand side of Eqs. (2.40) is expanded in the strength of the potential. This is suggested by the similar potential expansion used when eliminating the divergences in the density expansion of the equilibrium properties of the electron gas.⁽¹³⁾ The expansion in the potential strength is obtained as follows:

1. The Mayer functions appearing in the cluster expansion of the static correlations W_s (cf. Appendix A) are expanded in powers of $[-\beta V(r)]$.

2. The propagators $G_s(1 \cdots s, z)$ in Eqs. (2.40) are expanded in powers of θ_{ij} , by making repeated use of the operator identity,

$$G(1 \cdots s, z) = G_0(1 \cdots s, z) + G_0(1 \cdots s, z) \sum_{i \leq i < j \leq s} \theta_{ij} G(1 \cdots s, z) \quad (3.1)$$

where $G_0(1, \dots, s, z) = [z + \sum_{i=1}^s L_0(i)]^{-1}$ is the free s -particle propagator.

The interaction between two electrons governed by the operator θ_{ij} will in the following be referred to as a *weak* binary collision, to be contrasted with the *strong* binary collision between two particles interacting through a strong short-range potential, for instance, hard spheres. A collision is defined here as strong when the velocities of the colliding pair of particles are correlated at the end of the collision, weak otherwise. A binary collision will lead to velocity correlations when the action of the binary collision operator for particles 1 and 2 on a function of \mathbf{v}_1 results in a function not only of \mathbf{v}_1 , but also of the relative velocity \mathbf{v}_{12} . This is the case in a hard sphere collision. In contrast, the action of θ_{12} on a function of \mathbf{v}_1 always leads to a function of \mathbf{v}_1 only. Since the action of the operator θ_{ij} does not result in velocity correlations, the corresponding binary process is classified as a weak collision. We will argue in the following that the kernel of the BGL operator can be interpreted as describing an effective strong

binary collision between electrons, in analogy to the Boltzmann operator for neutral gases.

In order to analyze the B_s and classify their leading divergences, it is convenient to introduce a Fourier representation. In such a representation the free-particle propagators $G_0(1, \dots, s, z)$ are simply functions instead of operators. Owing to the translational invariance of the equilibrium averages, the Fourier transform of $C(1z; 1')$ is defined

$$C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}_{1'}) = \int d\mathbf{r}_{11'} e^{-i\mathbf{k} \cdot \mathbf{r}_{11'}} C(1z; 1') \quad (3.2)$$

where $\mathbf{r}_{11'} = \mathbf{r}_1 - \mathbf{r}_{1'}$.

From Eq. (2.38), the following equation for $C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}_{1'})$ is obtained:

$$\left[z + i\mathbf{k} \cdot \mathbf{v}_1 - \sum_{s=2}^{\infty} n^{s-1} B_{s\mathbf{k}}(\mathbf{v}_1, z) \right] C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}_{1'}) = C_{\mathbf{k}}(\mathbf{v}_1; \mathbf{v}_{1'}) \quad (3.3)$$

where

$$B_{s\mathbf{k}}(\mathbf{v}_1, z) = \int d\mathbf{r}_{11'} e^{-i\mathbf{k} \cdot \mathbf{r}_{11'}} B_s(1z) e^{i\mathbf{k} \cdot \mathbf{r}_{11'}} \quad (3.4)$$

The expansion of the operators $B_{s\mathbf{k}}$ in the strength of the potential will now be analyzed in some detail. There are two sources of divergences in such an expansion in the limit of $z \rightarrow 0$ and $k \rightarrow 0$:

(1) Divergences arise in the limit $\mathbf{k} \rightarrow 0$, for fixed z , due to the long range of the potential. The same behavior is already present in the equilibrium density expansion.

(2) Another type of divergence, associated with purely dynamical effects, appears in the limit $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$, due to the singular behavior of the free-particle propagator in such a limit. In general both (1) and (2) occur in each term of the potential expansion of the $B_{s\mathbf{k}}$.

In the following the potential expansion of B_2 and B_3 will be examined explicitly. This will indicate how to formulate a general rule to extract the terms that are most divergent in the limit $z \rightarrow 0^+$ in each order in the density. The resummation of the leading divergences is then carried out in two steps: first the static divergences are removed by a resummation that upgrades the divergent static pair correlation functions to convergent Debye-Hückel pair correlation functions; secondly, the dynamical divergences are resummed by renormalizing the free-particle propagators.

Specifically, the potential expansion of $B_{2\mathbf{k}}$ leads to

$$\begin{aligned}
 & nB_{2\mathbf{k}}(\mathbf{v}_1, z) C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}'_1) \\
 &= \mathcal{V}_{\mathbf{k}}(\mathbf{v}_1) C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}'_1) + n \int d\mathbf{v}_2 \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \\
 &\quad \times \frac{1}{z + i(\mathbf{k} - \mathbf{q}) \cdot \mathbf{v}_1 + i\mathbf{q} \cdot \mathbf{v}_2} \{ \theta_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) + \bar{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \\
 &\quad + [\theta_{\mathbf{k} - \mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) + \bar{\theta}_{\mathbf{k} - \mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)] P_{12} \} \phi(v_2) C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}'_1) \\
 &\quad + n \int d\mathbf{v}_2 \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) [-\beta V_{\mathbf{k} - \mathbf{q}}] P_{12} \phi(v_2) C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}'_1) \\
 &\quad + \text{L.D.T.} \tag{3.5}
 \end{aligned}$$

where L.D.T. stands for less divergent terms, and

$$\begin{aligned}
 \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) &= \int d\mathbf{r}_{12} e^{-i\mathbf{q} \cdot \mathbf{r}_{12}} \theta(1, 2) \\
 &= \frac{1}{m} \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2} \right) \tag{3.6}
 \end{aligned}$$

with $\boldsymbol{\varepsilon}_{\mathbf{q}} = i\mathbf{q}V_q$. Here V_q is the Fourier transform of the Coulomb potential, given by

$$V_q = \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} V(r) = \frac{4\pi e^2}{q^2} \tag{3.7}$$

In Eq. (3.5) $\bar{\theta}_{\mathbf{q}}$ is a function, given by

$$\begin{aligned}
 \bar{\theta}_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) &= -\phi^{-1}(v_1) \phi^{-1}(v_2) \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \phi(v_1) \phi(v_2) \\
 &= \beta \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \mathbf{v}_{12} \tag{3.8}
 \end{aligned}$$

It is proportional to the rate of change in kinetic energy of a pair of particles in a weak collision and it arises when shifting static correlation functions through the propagators, since the kinetic energy is not a conserved quantity in a collision. In general, in the representation used here for every term classified as most divergent (and therefore retained) in the expansion of each $B_{s\mathbf{k}}$, there is an identical term with the first (from the right) interaction operator $\theta_{\mathbf{q}}$ replaced by the function $\bar{\theta}_{\mathbf{q}}$, as seen in Eq. (3.5). The terms containing the function $\bar{\theta}_{\mathbf{q}}$ are as divergent as the corresponding terms containing the operator $\theta_{\mathbf{q}}$ and should therefore be

kept in a consistent theory. We will, however, show in the following that their contribution to the collision operator vanishes in the limit $z \rightarrow 0^+$, in which limit we are ultimately interested.

The first term on the right-hand side of Eq. (3.5) is the Vlasov operator,⁽¹⁷⁾ which represents the average effect on a given electron due to the presence of the other particles. It is given by

$$\mathcal{V}_{\mathbf{k}}(\mathbf{v}_1) = n \int d\mathbf{v}_2 \boldsymbol{\varepsilon}_{\mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{v}_1} \phi(\mathbf{v}_1) P_{12} \quad (3.9a)$$

or, in real space,

$$\mathcal{V}(1) = n \int d2\theta_{12} (1 + P_{12}) \phi(v_2) = n \int d2\theta_{12} P_{12} \phi(v_2) \quad (3.9b)$$

The second equality follows, because the part without permutation operator on the right-hand side of Eq. (3.9b) vanishes when acting on a function of 1, due to the \mathbf{r}_2 integration. When acting on the unknown function $C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}_1')$ the Vlasov operator gives

$$\mathcal{V}_{\mathbf{k}}(\mathbf{v}_1) C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}_1') = \frac{n}{m} \boldsymbol{\varepsilon}_{\mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{v}_1} \phi(v_1) \int d\mathbf{v}_2 C_{\mathbf{k}}(\mathbf{v}_2, z; \mathbf{v}_1'). \quad (3.9c)$$

Thus $\mathcal{V}_{\mathbf{k}} C_{\mathbf{k}}$ is directly related to density fluctuations, since $\int d\mathbf{v}_2 \int d\mathbf{v}_1' C_{\mathbf{k}}(\mathbf{v}_2, t; \mathbf{v}_1')$ is the Laplace transform of the density-density correlation function. The Vlasov operator is instantaneous, i.e., z independent, and is the most singular one in the limit $\mathbf{k} \rightarrow 0$, since it behaves as $\omega_p/k\lambda_D \sim n/k$, where ω_p is the plasma frequency $\omega_p = (k_B T/m)^{1/2}/\lambda_D$. The singular behavior of the Vlasov operator is due to the long range of the Coulomb interaction and is an exact property of the electron gas.⁽¹⁸⁾

The second term on the right-hand side of Eq. (3.5) represents a sequence of two successive weak collisions between the same two particles. It is the simplest ring collision event. In the limit $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$ the integral over the internal wavevector, \mathbf{q} , is logarithmically divergent at small q , i.e., it is of order $n \ln q$ or nq^0 . This divergence is related to the $z \rightarrow 0^+$ behavior of the free propagator.

The criterion adopted here to identify the divergences in the cluster expansion is the following: in each order in the density, n , the divergences will be classified on the basis of the strength of the small q singularity appearing in the internal wave vector integration as $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$. In B_2 , for instance, the most divergent terms are those that behave as $n \ln q$ or nq^0 as $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$. The second term on the right-hand side of Eq. (3.5) is the only most divergent term of order n .

Finally, the third term on the right-hand side of Eq. (3.5) is instantaneous, i.e., z independent, and related to density fluctuations, as is the

Vlasov operator. This term should be consistently neglected compared to terms that are divergent in the limit $z \rightarrow 0^+$, since it will lead to higher-order plasma parameter corrections to the leading long-time behavior of the correlation function discussed here. For instance, in the hydrodynamic regime the renormalized form⁶ of the instantaneous operator in question gives plasma parameter corrections to the ideal gas speed of sound. Similar corrections can, however, arise from noninstantaneous operators that are less divergent in the limit $z \rightarrow 0^+$ and have therefore been neglected. Thus in the following we will group instantaneous terms with less divergent ones and neglect them.

It is convenient for the following discussion to introduce a diagrammatic representation of the various collision sequences.

The diagram rules are as follows:

(1) A diagram consists of vertical lines, horizontal bonds, and crosses. The vertical lines are labeled at their bottom with the particle labels 1, 2, ... The velocities of all particles, except the field point velocity \mathbf{v}_1 , are integrated over.

(2) To obtain the diagram corresponding to a given term in an equation, the factors as they appear in this term from left to right are represented in the diagram from top to bottom, respectively.

(3) A cross and vertical line segment at the bottom level of the diagram $\downarrow \mathbf{k}$, label the root point, i.e., the velocity \mathbf{v}_i and the wave vector \mathbf{k} of the unknown function, $C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}_i)$.

(4) A cross and a vertical line segment at the top level of the diagram, $\uparrow \mathbf{k}$, label the external velocity and wave vector, i.e., the Fourier transform of the field variable 1.

(5) There is a factor $n\phi(v_i)$ associated with every label i that is not a root point. The Maxwellian factors are all located to the right of all operators, i.e., at the bottom of the diagram.

(6) Horizontal bonds are of two types: (i) static bonds that represent $(-\beta V_q) = \text{---}$, as obtained from the weak potential expansion of the Mayer functions, and drawn as dashed lines; (ii) dynamical bonds or θ bonds representing $\theta_q(\mathbf{v}_i, \mathbf{v}_j) = \text{~~~~}$, drawn as wavy lines.

(7) The vertical positions of the θ bonds in the diagrams define the levels in the diagram. In time language the bottom level corresponds to the

⁶ As $\mathbf{k} \rightarrow 0$, this instantaneous term contains a logarithmic divergence for small \mathbf{q} , due the singular behavior of the Coulomb potential for $\mathbf{q} \rightarrow 0$. This divergence is removed when the factor $[-\beta V_{\mathbf{k}-\mathbf{q}}]$ is renormalized to a Debye-Hückel pair pair correlation function. On the contrary, the divergent behavior of the noninstantaneous terms persists even after the renormalization of the equilibrium correlation functions.

smallest time; an ordered time integration, $\int_0^t dt_1 \int_{t_1}^t dt_2 \int_{t_2}^t dt_3 \dots$, with $t \geq t_1 \geq t_2 \geq \dots$, is to be performed over the times of all intermediate levels. The same diagrammatic representation is used for equations in time and in Laplace language.

(8) Internal vertical line segments between the diagram levels represent free propagation of the particles, i.e., $\mathbf{q} \uparrow_0' = \exp(-i\mathbf{q} \cdot \mathbf{v}_i t)$, or $\mathbf{q} \uparrow_1 \downarrow_2 \mathbf{q}' = [z + i\mathbf{q} \cdot \mathbf{v}_1 + i \cdot \mathbf{q}' \cdot \mathbf{v}_2]^{-1}$.

(9) Wave vectors are conserved at each vertex and all internal wave vectors are integrated over.

The collision operators in Eq. (3.6) may be represented then as in Fig. 1. The diagrams containing the function $\bar{\theta}_\mathbf{q}$ have been omitted in Fig. 1. They are identical to the last two diagrams, with the bottom $\theta_\mathbf{q}$ bond replaced by $\bar{\theta}_\mathbf{q}$. The first of the diagrams in Fig. 1 represents the Vlasov operator $\mathcal{V}_\mathbf{k}'(\mathbf{v}_1)$. This completes the discussion of $B_{2\mathbf{k}}$.

A similar analysis may be carried out for $B_{3\mathbf{k}}$. It is easy to see that the most divergent terms are those where the internal wave vector integration contains a small \mathbf{q} divergence of order $n^2 e^2 / q^2$ in the limit $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$. They may be divided into two groups:

(1) Purely dynamical diagrams, having the structure of ring diagrams and representing three collisions among three particles. There are eight nonvanishing diagrams of this kind, given by the four diagrams of Fig. 2a, plus the same four diagrams with the bottom θ bond replaced by $\bar{\theta}$.

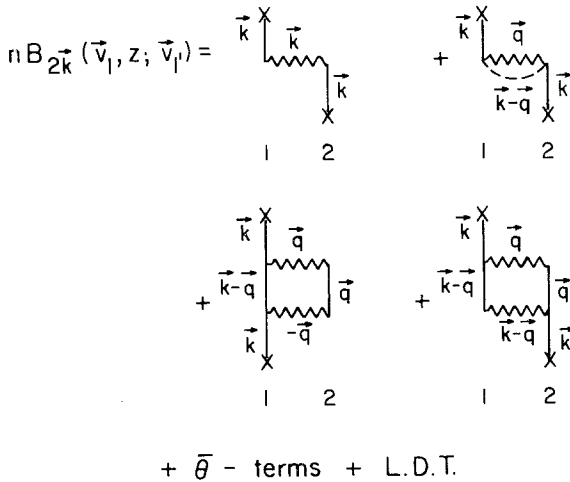


Fig. 1. The potential expansion of B_2 . The first diagram represents the Vlasov operator, $\mathcal{V}_\mathbf{k}'(\mathbf{v}_1)$.

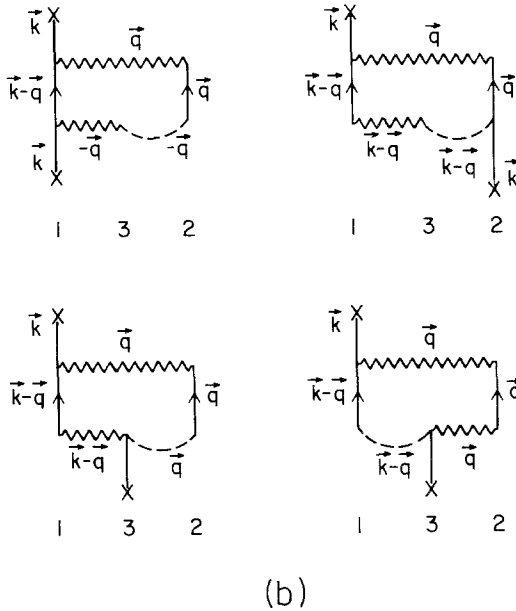
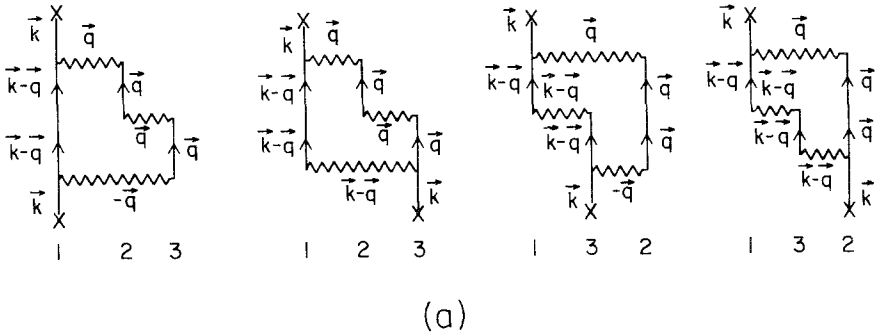


Fig. 2. The most divergent diagrams in B_3 : (a) purely dynamical ring diagrams; (b) diagrams with two interaction bonds and one static bond.

(2) Three-particle diagrams containing dynamical bonds and one static bond. There are eight such diagrams, given by the four diagrams of Fig. 2b and the same four diagrams with the bottom θ bond replaced by $\bar{\theta}$. These diagrams have a new structure not found in B_2 .

The last two diagrams of Fig. 2b have the structure of corrections to the Vlasov operator, since when acting on $C_k(\mathbf{v}_1, z; \mathbf{v}_1')$ they are related to the density-density correlation function. It is, however, easy to show that

the sum of these two diagrams together with the corresponding two diagrams where the bottom $\theta_{\mathbf{q}}$ bond has been replaced by the function $\bar{\theta}_{\mathbf{q}}$, leads to terms that are less divergent in the limit $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$ than those retained here.

All other terms in $B_{3\mathbf{k}}$ are less divergent, in the sense that the \mathbf{q} integration leads to a less singular result at small \mathbf{q} when $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$.

A similar systematic analysis can be carried out for the higher-order collision operators. This has been done in detail only for $B_{4\mathbf{k}}$ and $B_{5\mathbf{k}}$. We have then formulated a rule to collect the most divergent terms in each order in the density: the most divergent terms in $B_{s\mathbf{k}}$ are those where the internal \mathbf{q} integration contains a small \mathbf{q} singularity of order $n(ne^2/q^2)^s$ in the limit $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$.

These terms are represented by the following diagrams:

(1) All diagrams containing s dynamical bonds $\theta_{\mathbf{q}}$ and no static potential bonds. These correspond to sequences of s weak binary collisions among s particles. They have the same structure as the dynamical ring events that determine the leading correction to the Boltzmann equation for neutral gases. There the binary collisions are, however, strong collisions.

(2) All diagrams containing $(s-k)$ interaction bonds θ_{ij} and k static potential bonds, for $k = 1, 2, \dots, s-2$.

We emphasize that, in contrast to the case of a gas with strong short-range forces, terms containing static correlations are among the most divergent contributions in each order in the density. They are needed for a complete cancellation of all the terms represented by singly connected diagrams and they will appear explicitly in the kinetic equation for the electron gas, even to lowest order in the plasma parameter, ϵ_p .

The resummation of these terms can be carried out in two steps.

First the purely static divergences are eliminated by resumming the chain Mayer graphs, as done in the equilibrium virial expansion. To exemplify this, we display such resummation for the collision sequence corresponding to the first diagram in Fig. 2b, i.e.,

$$\begin{aligned}
 & n \int d\mathbf{v}_2 \int d\mathbf{v}_3 \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \frac{1}{z + i(\mathbf{k} - \mathbf{q}) \cdot \mathbf{v}_1 + i\mathbf{q} \cdot \mathbf{v}_2} \theta_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_3) \\
 & \times \sum_{l=1}^{\infty} \left\{ n \int d\mathbf{v}_4 [-\beta V_q] \phi(v_4) \right\}^l \phi(v_2) \phi(v_3) C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}_1) \\
 & = n \int d\mathbf{v}_2 \int d\mathbf{v}_3 \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \frac{1}{z + i(\mathbf{k} - \mathbf{q}) \cdot \mathbf{v}_1 + i\mathbf{q} \cdot \mathbf{v}_2} \\
 & \times \theta_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_3) h_{\text{DH}}(q) \phi(v_2) \phi(v_3) C_{\mathbf{k}}(\mathbf{v}_1, z; \mathbf{v}_1) \quad (3.10)
 \end{aligned}$$

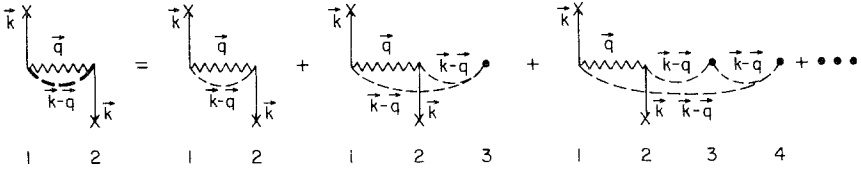


Fig. 3. Resummation of the static Mayer functions to a Debye-Hückel pair correlation.

The diagrammatic representation of the resummation is displayed in Fig. 3. In Eq. (3.10)

$$h_{\text{DH}}(q) = \frac{1}{n} \sum_{l=1}^{\infty} [-n\beta V_q]^l = -\frac{1}{n} \frac{1}{1 + q^2 \lambda_D^2} \quad (3.11)$$

is the Debye-Hückel pair correlation function, represented in Fig. 3 by a thick dashed line. This resummation replaces all the static correlations with Debye-Hückel pair correlation functions, that are finite at large interparticle separation.

The dynamical divergences are eliminated by summing the ring diagrams. Again, it is convenient to display this resummation for the simplest case, where dynamical and static divergences do not appear simultaneously. The sum of all ring diagrams can most easily be written in real space and reads

$$\begin{aligned} & \sum_{l=0}^{\infty} n \int d^2\theta_{12} G_0(12, z) [n(1 + P_{12}) \int d^3\theta_{13} (1 + P_{13}) \phi(v_3) G_0(12, z)]^l \\ & \quad \times \theta_{12}(1 + P_{12}) \phi(v_2) \\ & = n \int d^2\theta_{12} G_v(12, z) \theta_{12}(1 + P_{12}) \phi(v_2) \end{aligned} \quad (3.12a)$$

where $G_v(12, z)$ is a two-particle Vlasov propagator,

$$G_v(12, z) = [z + L_v(12)]^{-1} \quad (3.12b)$$

with $L_v(12) = L_v(1) + L_v(2)$, and $L_v(1) = L_0(1) - \mathcal{V}(1)$.

By resumming all the terms from each $B_{s\mathbf{k}}$ that are of order $n(ne^2/q^2)^s$ when $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$ and using Eqs. (3.10) and (3.12), one obtains the desired generalized (inhomogeneous and non-Markoffian) BGL collision operator, given by

$$\begin{aligned} A_{\mathbf{k}}(\mathbf{v}_1, z) & = n \int d\mathbf{v}_2 \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) G_v(\mathbf{k} - \mathbf{q}, \mathbf{v}_1; \mathbf{q}, \mathbf{v}_2; z) \\ & \quad \times \{ \hat{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) + \hat{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) + [\hat{\theta}_{\mathbf{q}-\mathbf{k}}(\mathbf{v}_2, \mathbf{v}_1) \\ & \quad + \hat{\theta}_{\mathbf{q}-\mathbf{k}}(\mathbf{v}_2, \mathbf{v}_1)] P_{12} \} \phi(v_2) \end{aligned} \quad (3.13a)$$

where

$$\hat{\theta}_q(\mathbf{v}_1, \mathbf{v}_2) = \theta_q(\mathbf{v}_1, \mathbf{v}_2) + n \int d\mathbf{v}_3 \theta_q(\mathbf{v}_1, \mathbf{v}_3) h_{\text{DH}}(q) \phi(v_3) \quad (3.13b)$$

or, using Eqs. (3.6),

$$\hat{\theta}_q(\mathbf{v}_1, \mathbf{v}_2) = \frac{1}{m} S_{\text{DH}}(q) \boldsymbol{\varepsilon}_q \cdot \frac{\partial}{\partial \mathbf{v}_1} - \frac{1}{m} \boldsymbol{\varepsilon}_q \cdot \frac{\partial}{\partial \mathbf{v}_2} \quad (3.13c)$$

The function $\hat{\theta}_q$ is defined by Eq. (3.13b), with each θ_q replaced by a $\hat{\theta}_q$. In Eq. (3.13b) $S_{\text{DH}}(q)$ is the Debye–Hückel approximation for the static structure factor, $S(q)$, which is defined by

$$S(q) = 1 + nh_2(q) \quad (3.14a)$$

while $S_{\text{DH}}(q)$ is given by

$$S_{\text{DH}}(q) = \frac{q^2 \lambda_D^2}{1 + q^2 \lambda_D^2} \quad (3.14b)$$

The diagrammatic representation of the four terms occurring in Eq. (3.13) is given in Fig. 4, where the thick vertical lines denote the renormalized

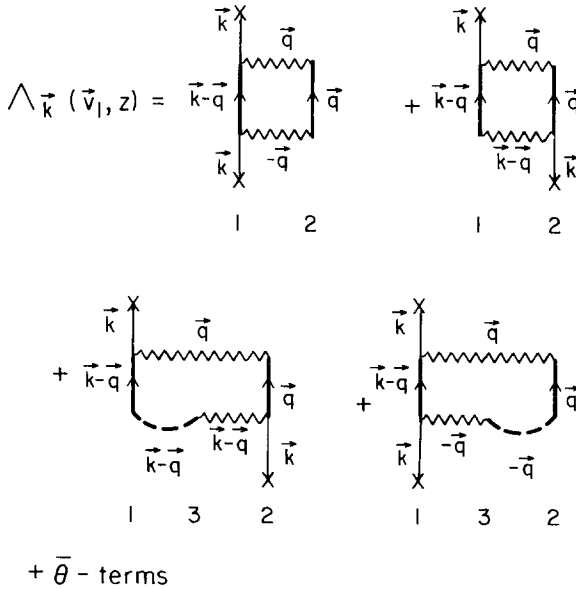


Fig. 4. Diagrammatic representation of Eq. (3.13).

propagators. In writing Eq. (3.13) a new dynamical operator $\hat{\theta}$ containing the static correlations has been defined.

The corresponding kinetic equation for $C_{\mathbf{k}}(\mathbf{v}_1 z; \mathbf{v}_1')$ is then given by

$$[z + i\mathbf{k} \cdot \mathbf{v}_1 - \mathcal{V}_{\mathbf{k}}(\mathbf{v}_1) - A_{\mathbf{k}}(\mathbf{v}_1, z)] C_{\mathbf{k}}(\mathbf{v}_1 z; \mathbf{v}_1') = C_{\mathbf{k}}(\mathbf{v}_1; \mathbf{v}_1') \quad (3.15)$$

An explicit representation of $A_{\mathbf{k}}(\mathbf{v}_1, z)$ in the limit $\mathbf{k} \rightarrow 0$ and $z \rightarrow 0^+$ can be obtained following closely the procedure used in Ref. 17 for the nonlinear BGL operator in the spatially homogeneous case. The details of this reduction are given in Appendix B. In this limit the generalized operator obtained here is found to be identical to the linearized and Markoffian BGL operator for the homogeneous electron gas used in the literature, i.e.,

$$\begin{aligned} A_0(\mathbf{v}_1) &= \lim_{\substack{\mathbf{k} \rightarrow 0 \\ z \rightarrow 0^+}} A_{\mathbf{k}}(\mathbf{v}_1, z) \\ &= -\frac{\pi}{m^2} \int \frac{d\mathbf{q}}{(2\pi)^3} \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{v}_1} \frac{[V_q]^2}{|D(-\mathbf{q}, i\mathbf{q} \cdot \mathbf{v}_1)|^2} \int d\mathbf{v}_2 \delta(\mathbf{q} \cdot (\mathbf{v}_1 - \mathbf{v}_2)) \\ &\quad \times \mathbf{q} \cdot \left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2} \right) (1 + P_{12}) f_1(v_2) \end{aligned} \quad (3.16a)$$

where

$$D(\mathbf{q}, z) = 1 - \int d\mathbf{v} \frac{1}{z + i\mathbf{q} \cdot \mathbf{v}} \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}} f_1(\mathbf{v}) \quad (3.16b)$$

is the plasma dispersion function, defined for $\text{Re}z > 0$.

The terms containing static bonds in Eq. (3.13) are needed to obtain the form given in Eq. (3.16). This point has been noticed before by Krommes and Oberman.⁽⁶⁾

The part of the operator $A_{\mathbf{k}}(\mathbf{v}_1, z)$ containing the function $\hat{\theta}$ leads to a vanishing contribution in the limit $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$, because $\delta(\mathbf{q} \cdot \mathbf{v}_{12}) \hat{\theta}_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}) = 0$ [cf. Eq. (3.8)]. It was, however, essential to retain the terms containing the function $\hat{\theta}$, as was argued before when discussing the last two diagrams of Fig. 2b.

4. THE INHOMOGENEOUS LINEARIZED BGL EQUATION FROM THE BBGKY HIERARCHY

Guernsey⁽⁵⁾ and Krommes and Oberman⁽⁶⁾ have derived a linearized BGL equation for an inhomogeneous electron gas by truncating the

BBGKY hierarchy at the level of the second hierarchy equation. They obtained the following kinetic equation:

$$[z + L_v(1) - \tilde{A}(1, z) - \delta\mathcal{V}^{(s)}(1, z)] C(1, z; 1') = C(1; 1') + I(1z; 1') \quad (4.1)$$

which differs from Eq. (3.13) in the presence of a Vlasov correction term, given by

$$\delta\mathcal{V}^{(s)}(1', z) = n(1 + P_{12}) \int d^3\theta_{13} \phi(v_1) \phi(v_2) h_{\text{DH}}(r_{12}) P_{13} \quad (4.2a)$$

and an initial condition correction term, given by

$$I(1, z; 1') = \int d^2\theta_{12} \frac{1}{z + L_v(12)} C(12; 1') \quad (4.2b)$$

Also $\tilde{A}(1, z)$ differs from the linearized BGL operator obtained from the cluster expansion [cf. Eq. (3.13a)] for it does not contain the terms in which the operator θ is replaced by the function $\bar{\theta}$, i.e.,

$$\begin{aligned} \tilde{A}(1, z) = n \int d^2\theta_{12} \frac{1}{z + L_v(12)} (1 + P_{12}) \\ \times \left[\theta_{12} + n \int d^3\theta_{13} h_{\text{DH}}(r_{23}) \phi(v_3) \right] \phi(v_2) \end{aligned} \quad (4.2c)$$

The diagrammatic representation of the Fourier transform of Eq. (4.2a) is given in Fig. 5: $\delta\mathcal{V}^{(s)}$ contains singly connected diagrams and is as singular as the Vlasov operator itself at large distances, since $\delta\mathcal{V}_{\mathbf{k}}^{(s)}(\mathbf{v}_1, z) \sim n^{3/2}/k$ for $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$. Both operators $\delta\mathcal{V}^{(s)}$ and I are then neglected in Ref. 6, where it is argued that they represent corrections to the Vlasov operator and to the initial condition $C(1; 1')$, respectively. Although this statement is correct, the appearance in Eq. (4.1) of singular corrections to the Vlasov

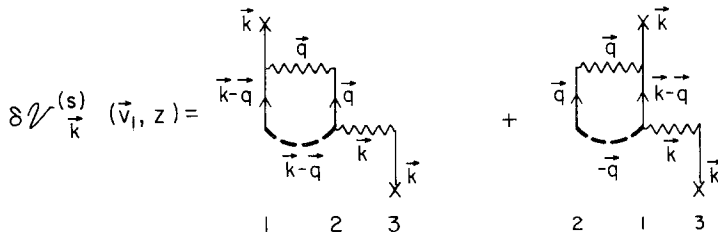


Fig. 5. Diagrammatic representation of $\delta\mathcal{V}^{(s)}$, Eq. (4.2).

operator is somewhat disconcerting. Such terms, if not canceled to higher order, could lead to unphysical behavior.

The objective of this section is to clarify the connection between the cluster expansion method, where singular Vlasov corrections never appear, and the hierarchy method. We show here that the singular operator $\delta\mathcal{V}^{(s)}$ is actually canceled by part of $I(1, z; 1')$. It is then possible to recover the results of the cluster expansion by using a consistent closure of the hierarchy. To do this, it is convenient to characterize the various terms occurring in the hierarchy by introducing dimensionless variables. Defining $z^* = z/\omega_p$, $\mathbf{r}^* = \mathbf{r}/\lambda_D$ and $\mathbf{v}^* = \mathbf{v}/v_0$, with $v_0 = (m\beta)^{-1/2}$, dimensionless distributions and correlation functions are given by

$$\phi^*(v_1^*) = (v_0^3/n) f_1(1) \quad (4.3a)$$

$$G_s^*(1^*, \dots, s^*) = (v_0^{3s} \lambda_D^{3(s-1)}/n) G_s(1, \dots, s) \quad (4.3b)$$

$$C^*(1^*, \dots, s^*; 1'^*) = (v_0^{3(s+1)} \lambda_D^{3s}/n) C(1, \dots, s; 1') \quad (4.3c)$$

$$C^*(1^*, \dots, s^*, z^*; 1'^*) = (\omega_p v_0^{3(s+1)} \lambda_D^{3s}/n) C(1, \dots, s, z; 1') \quad (4.3d)$$

where $C(1, \dots, s, z; 1')$ is the Laplace transform of the two-time correlation function

$$C(1, \dots, s, t; 1') = \langle \delta\psi(1, \dots, s, t) \delta\psi(1', 0) \rangle \quad (4.4)$$

In terms of the dimensionless quantities defined above the first two equations of the hierarchy for the Laplace transforms of the two time equilibrium correlation functions are (the stars on the reduced variables are suppressed to simplify the notation):

$$[z + L_v^*(1)] C^*(1, z; 1') = C^*(1; 1') + \varepsilon_p \int d2\theta_{12}^* C^*(12z; 1') \quad (4.5a)$$

and

$$\begin{aligned} [z + L_v^*(12) - \varepsilon_p \theta_{12}^*] C^*(12z; 1') &= C^*(12; 1') \\ &+ \frac{1}{4\pi} \theta_{12}^*(1 + P_{12}) \phi^*(v_2) C^*(1z; 1') \\ &+ \frac{1}{4\pi} (1 + P_{12}) \int d3\theta_{13}^*(1 + P_{13}) G_2^*(23) C^*(1z; 1') \\ &+ \varepsilon_p (1 + P_{12}) \int d3\theta_{13}^* C^*(123z; 1') \end{aligned} \quad (4.5b)$$

where

$$L_v^*(1) = L_0(1) - \frac{1}{4\pi} \mathcal{V}^*(1),$$

with

$$\mathcal{V}^*(1) = \int d^3\theta_{13}^* \phi^*(v_1) P_{13} \quad (4.5c)$$

and

$$\theta_{12}^* = \frac{1}{e^2} \frac{\partial V(r_{12})}{\partial \mathbf{r}_1} \cdot \left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2} \right) \quad (4.5d)$$

Guernsey⁽⁵⁾ and Krommes and Oberman⁽⁶⁾ truncated Eq. (4.5b) by neglecting the term $\varepsilon_p \theta_{12}^* C^*(12z; 1')$ on the left-hand side of the equation, as well as the three particle cluster function $\varepsilon_p C^*(123z; 1')$ on the right-hand side of the equation, consistently⁽¹⁹⁾ with their ordering in the plasma parameter. There is no reason at this point to neglect the initial condition in the second equation, $C^*(12; 1')$. In fact, we will show that this term cancels the singular Vlasov correction (4.2) on the left-hand side of Eq. (4.1). Neglecting the two terms of $O(\varepsilon_p)$ in Eq. (4.5b), solving for $C^*(12z; 1')$ and substituting in Eq. (4.5a), the following equation for $C^*(1z; 1')$ is obtained:

$$\begin{aligned} & \left[z + L_v^*(1) - \frac{\varepsilon_p}{4\pi} \tilde{A}^*(1, z) - \frac{\varepsilon_p}{4\pi} \delta \mathcal{V}^{(s)*}(1, z) + O(\varepsilon_p^2) \right] C^*(1z; 1') \\ & = C^*(1; 1') + \varepsilon_p I^*(1z; 1') \end{aligned} \quad (4.6a)$$

with

$$I^*(1z; 1') = \int d^2\theta_{12}^* \frac{1}{z + L_v^*(12)} C^*(12; 1') \quad (4.6b)$$

Since our objective is to obtain a closed kinetic equation for $C^*(1z; 1')$, we need to express the second term on the right-hand side of Eq. (4.6) in terms of the unknown function $C^*(1z; 1')$. This can be done by using Eq. (2.24) to explicitly introduce the initial condition $C^*(1; 1')$ into Eq. (4.6b), i.e.,

$$\begin{aligned} \varepsilon_p I^*(1z; 1') & = \varepsilon_p \int d^2d1'' d^3\theta_{12}^* \frac{1}{z + L_v^*(12)} \\ & \times C^*(12; 1'') C^{*-1}(1''; 3) C^*(3; 1') \end{aligned} \quad (4.7a)$$

Using then Eq. (4.6a) to lowest order in ε_p we can express $I^*(1z; 1')$ in terms of the unknown function $C^*(1z; 1')$ as

$$\begin{aligned} \varepsilon_p I^*(1z; 1') & = \varepsilon_p \int d^2d^3\theta_{12}^* \frac{1}{z + L_v^*(12)} \\ & \times \int d1'' C^*(12; 1'') C^{*-1}(1''; 3) \\ & \times [z + L_v^*(3)] C^*(3z; 1') + O(\varepsilon_p^2) \end{aligned} \quad (4.7b)$$

For consistency, the equal time correlations in Eq. (4.7b) should be evaluated to lowest order in the plasma parameter. To this order, it is easy to express the equal time correlation functions in Eq. (4.7b) in terms of equilibrium distribution functions. Using Eqs. (2.23) and (2.25), together with the convolution approximation—which is consistent with the Debye–Hückel approximation—for the three-particle cluster function $h_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, given by⁽¹⁹⁾

$$h_{\text{DH}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = h_{\text{DH}}(r_{12}) h_{\text{DH}}(r_{13}) + h_{\text{DH}}(r_{12}) h_{\text{DH}}(r_{23}) + h_{\text{DH}}(r_{13}) h_{\text{DH}}(r_{32}) \\ + n \int d\mathbf{r}_4 h_{\text{DH}}(r_{14}) h_{\text{DH}}(r_{24}) h_{\text{DH}}(r_{34}) \quad (4.8)$$

one finds

$$\int d1'' C^*(12; 1'') C^{*-1}(1''; 3) \\ = (1 + P_{12}) \phi^*(v_2) h_{\text{DH}}^*(r_{12}) \delta(1-3) \\ + \phi^*(v_2) \phi^*(v_3) h_{\text{DH}}^*(r_{13}) h_{\text{DH}}^*(r_{32}) + O(\varepsilon_p) \quad (4.9)$$

with $h_{\text{DH}}^*(r_{12}^*) = n\lambda_{\text{D}}^3 h_{\text{DH}}(r_{12})$.

Substituting Eq. (4.9) into Eq. (4.7b) and writing the inverse propagator in Eq. (4.7) as the sum of a free propagator and a Vlasov operator, $I^*(1z; 1')$ can be rewritten

$$\varepsilon_p I^*(1z; 1') = -\frac{\varepsilon_p}{4\pi} \delta v^{(s)*}(1, z) C^*(1z; 1') \\ + \varepsilon_p \int d2\theta_{12}^* \frac{1}{z + L_v^*(12)} (1 + P_{12}) \phi^*(v_2) h_{\text{DH}}^*(r_{12}) \\ \times [z + L_0(12) C^*(1z; 1')] \\ + \varepsilon_p \int d2d3\theta_{12}^* \frac{1}{z + L_v^*(12)} \phi^*(v_2) \phi^*(v_3) h_{\text{DH}}^*(r_{13}) h_{\text{DH}}^*(r_{32}) \\ \times [z + L_0(123)] C^*(1z; 1') + O(\varepsilon_p^2) \quad (4.10)$$

When Eq. (4.10) is inserted in Eq. (4.6a), the first term on the right-hand side cancels the singular Vlasov correction term in Eq. (4.6a). The other two terms in Eq. (4.10) can be rewritten in a more convenient form by shifting the inverse free propagators to the left through the static correlation functions. This can be done by using the equations of the equilibrium hierarchy to lowest order in the plasma parameter. The

calculation is tedious, but straightforward and will not be given here. As a result, one finds that the second two terms in Eq. (4.10) contain those terms contributing to the BGL operator in which the last θ operator is replaced by a $\bar{\theta}$ function [cf. Eq. (3.13a)]. These contributions were also obtained from the cluster expansion. In addition, they also contain terms that do not contribute to leading order as $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$ and should therefore be neglected. The equation obtained from the hierarchy is then identical to that obtained from the cluster expansion.

In contrast to the hierarchy method, the cluster expansion directly provides a closed equation for $C(1z; 1')$, i.e., an equation that determines its evolution from a given initial condition $C(1; 1')$. This is achieved when transforming from Eq. (2.33) to Eq. (2.30) using the standard inversion procedure,⁽¹⁵⁾ which eliminates all singly connected diagrams contained in the $\delta\mathcal{V}^{(s)}$. In the hierarchy truncation, however, only part of the singly connected diagrams are automatically canceled. Thus, after closing the hierarchy, a further rearrangement of the resulting kinetic equation is needed to complete the elimination of the naive density divergence.

It is not clear how the hierarchy method can consistently be extended to higher orders in the plasma parameter. It will become increasingly more complicated to express the equal time correlation functions arising from the initial condition for the higher-order equations of the hierarchy in terms of the unknown function and identify that part of the static correlations that needs to be kept to construct a consistent kinetic equation for $C(1z; 1')$.

Finally, the elimination of the singly connected diagrams and of the divergences occurring in the naive density expansion is also automatically achieved by using the algebraic methods mentioned in Section 1.⁽⁷⁻⁹⁾ As discussed in the Introduction, such theories do not, however, contain a natural small parameter and are therefore not systematic. The cluster expansion method is then more reliable since the approximations introduced there can in general be better controlled.

5. DISCUSSION

We conclude the paper with a number of remarks.

(1) There is a close formal analogy between the derivation of the BGL operator given here for an electron gas in a uniform positive background and that of the ring kinetic operator for neutral gases with strong short-range interactions. In both cases the collision operator is obtained by summing the most divergent contributions due to ring collisions and has the structure of two successive binary collisions separated by a renormalized propagator that includes the effect of

collisional damping in the case of neutral gases and dynamical screening in the case of the electron gas. There are, however, important differences.

(a) In the neutral gas case the ring operator is constructed by summing sequences of strong collisions. The intermediate propagator in the resulting kinetic operator contains long-lived hydrodynamic fluctuations. This is in contrast to the Boltzmann equation which only describes fluctuations with a lifetime $\sim \tau_c^B$, where τ_c^B is the Boltzmann mean free time. The BGL operator is constructed by sequences of weak collisions. The intermediate renormalized propagator contains the effect of dynamical screening due to the other particles and, in a gross sense, provides an effective cutoff of the Coulomb interaction at large distances. The BGL equation describes the decay of fluctuations of lifetime $\sim \tau_c$. In this sense it contains the same physical information as the Boltzmann equation. Its kernel can be interpreted as an effective binary collision operator, representing a strong collision.

(b) Another important difference between neutral and charged gases lies in the role played by the static correlation functions. In neutral gases they do not contribute to the most divergent collision sequences, while for systems with long-range potentials the static correlation functions do contribute to leading order in the plasma parameter, so that the inhomogeneous BGL operator itself explicitly contains the two-particle correlation function. This is because the Debye-Hückel pair correlation function,

$$G_2^{\text{DH}}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}) = n^2 \phi(v_1) \phi(v_2) h_{\text{DH}}(q)$$

is, for small $q (q \ll \lambda_D^{-1})$, of the same order in the density as the single-particle distribution function, $f_1(1) = n\phi(v_1)$, since $h_{\text{DH}}(q) \sim n^{-1}$ [cf. Eq. (3.14)].

(2) The Markovian form of the BGL collision operator cannot consistently be used to obtain the hydrodynamic equations for the electron gas. In the hydrodynamic regime the electron gas can sustain finite frequency plasma modes that can only be described through the use of a frequency-dependent collision operator, to be evaluated at a frequency ω , with $\omega = iz$, of the order of the plasma frequency, ω_p . To describe long-wavelength phenomena in a dilute electron gas one must, however, use the $\mathbf{k} \rightarrow 0$ limit of the collision operator. This is because its \mathbf{k} dependence leads to plasma parameter corrections to the dilute gas expressions for the transport coefficients and the thermodynamic quantities that appear in the hydrodynamic equations. These corrections are of the same order in the plasma parameter as those resulting from terms that have been neglected in our derivation (e.g., less divergent terms in the cluster expansion) and should then be ignored.

(3) In the Markovian limit, i.e., $z \rightarrow 0^+$, the same representation for the collision operator as given in Eq. (3.16) is obtained when all the terms containing the functions $\hat{\theta}_q$ are neglected in Eq. (3.13). This is due to the fact that the Markovian limit of the BGL operator conserves kinetic energy, because of the presence of the δ function, $\delta(\mathbf{q} \cdot \mathbf{v}_{12})$, in Eq. (3.16).

(4) The same method used here may be employed to study the problem of tagged particle motion. The quantity of interest is then the correlation function of fluctuations of the phase space density of the tagged particle, which is defined

$$C_s(1t; 1') = \langle \delta(1 - X_1(t)) [\delta(1' - X_1) - \langle \delta(1' - X_1) \rangle] \rangle \quad (5.1)$$

where X_1 denotes here the phase of the tagged particle. Using the cluster expansion method, the "self"-BGL equation, describing tagged particle motion, is found to be

$$[z + i\mathbf{k} \cdot \mathbf{v}_1 - A_{sk}(\mathbf{v}_1, z)] C_{sk}(\mathbf{v}_1, z; \mathbf{v}_1') = C_{sk}(\mathbf{v}_1; \mathbf{v}_1') \quad (5.2a)$$

with

$$\begin{aligned} A_{sk}(\mathbf{v}_1, z) = n \int d\mathbf{v}_2 \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_q(\mathbf{v}_1, \mathbf{v}_2) \frac{1}{z + i(\mathbf{k} - \mathbf{q}) \cdot \mathbf{v}_1 + i\mathbf{q} \cdot \mathbf{v}_2 - \mathbf{q}(\mathbf{v}_2)} \\ \times [\hat{\theta}_q(\mathbf{v}_1, \mathbf{v}_2) + \hat{\hat{\theta}}_q(\mathbf{v}_1, \mathbf{v}_2)] \phi(v_2) \end{aligned} \quad (5.2b)$$

A different representation of A_s for $\mathbf{k} = 0$ was given in Ref. 4. The two representations are identical in the limit $z \rightarrow 0^+$, where they both reduce to

$$\begin{aligned} A_{s0}(\mathbf{v}_1) = -\frac{\pi}{m^2} \int \frac{d\mathbf{q}}{(2\pi)^3} \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{v}_1} \frac{V_q^2}{|D(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)|^2} \delta(\mathbf{q} \cdot (\mathbf{v}_1 - \mathbf{v}_2)) \\ \times \mathbf{q} \cdot \left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2} \right) n\phi(v_2) \end{aligned} \quad (5.3)$$

(5) Finally, Dorfman and Cohen⁽¹²⁾ have used the methods of the nonequilibrium cluster expansion to derive a generalized (inhomogeneous and non-Markoffian) form of the *nonlinear* BGL operator. They showed that the nonlinear operator is also obtained by summing ring collisions, as done in the homogeneous case by Balescu.⁽³⁾ The linearization of their kinetic equation does not, however, reproduce Eq. (3.13) obtained here. It contains the linear BGL operator $\tilde{A}(1, z)$ defined in Eq. (4.2c) and the singular Vlasov correction as obtained when closing the hierarchy. We will discuss the resolution of this point in a separate publication.⁽²⁰⁾

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APPENDIX A: CLUSTER EXPANSION OF THE EQUILIBRIUM CORRELATION FUNCTION

In this Appendix the explicit form of the cluster expansion of the $W_s(1, \dots, s)$, as defined in Eq. (2.28), is obtained.

The equilibrium virial expansion of the reduced distribution functions $g_s(\mathbf{r}_1, \dots, \mathbf{r}_s)$ is discussed for example in Ref. 21. It has the form

$$g_s(\mathbf{r}_1, \dots, \mathbf{r}_s) = \sum_{l=0}^{\infty} n^l g_s^{(l)}(\mathbf{r}_1, \dots, \mathbf{r}_s) \quad (\text{A1})$$

where

$$g_s^{(l)}(\mathbf{r}_1, \dots, \mathbf{r}_s) = \int d\mathbf{r}_{s+1} \cdots d\mathbf{r}_{s+l} g_s(\mathbf{r}_1, \dots, \mathbf{r}_s | \mathbf{r}_{s+1}, \dots, \mathbf{r}_{s+l}) \quad (\text{A2})$$

The first term, corresponding to $l=0$, is simply given by

$$\begin{aligned} g_s^{(0)}(\mathbf{r}_1, \dots, \mathbf{r}_s) &= \exp \left[-\beta \sum_{1 \leq i < j \leq N} V(r_{ij}) \right] \\ &= \prod_{1 \leq i < j \leq N} (1 + f_{ij}) \end{aligned} \quad (\text{A3})$$

where $f_{ij} = e^{-\beta V(r_{ij})} - 1$ is the Mayer f function, and for $l=1$,

$$\begin{aligned} g_s^{(1)}(\mathbf{r}_1, \dots, \mathbf{r}_s) &= g_s^{(0)}(\mathbf{r}_1, \dots, \mathbf{r}_s) \int d\mathbf{r}_{s+1} \sum_{j=2}^s \\ &\quad \times \sum_{1 \leq i_1 < \cdots < i_j \leq N} f_{i_1, s+1} \cdots f_{i_j, s+1} \end{aligned} \quad (\text{A4})$$

From Eq. (A1) and the definition of the cluster functions h_s given in Eqs. (2.22) the virial expansion of the h_s may also be obtained in a similar form, i.e.,

$$h_s(\mathbf{r}_1, \dots, \mathbf{r}_s) = \sum_{l=0}^{\infty} n^l h_s^{(l)}(\mathbf{r}_1, \dots, \mathbf{r}_s) \quad (\text{A5})$$

The explicit expression for the first few terms in Eq. (A5) for $s=2, 3$ may again be found in Ref. 21. Finally, the cluster expansion for the direct correlation function will be needed. It is given by

$$\begin{aligned} c(\mathbf{r}_{12}) &= \sum_{l=0}^{\infty} n^l c_l(\mathbf{r}_1, \mathbf{r}_2) \\ &= f_{12} + \sum_{s=3}^{\infty} \frac{n^{s-1}}{(s-2)!} \int d\mathbf{r}_3 \cdots d\mathbf{r}_s B(\mathbf{r}_1, \dots, \mathbf{r}_s) \end{aligned} \quad (\text{A6})$$

where $B(\mathbf{r}_1, \dots, \mathbf{r}_s)$ is the sum of all biconnected graphs of s particles. In particular

$$B(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = f_{12} f_{23} f_{31} \quad (\text{A7})$$

In order to construct the equilibrium virial expansion for the W_s , the latter need to be expressed entirely in terms of equilibrium distribution functions. Combining Eqs. (2.28) and (2.17), the first few W_s are explicitly

$$\begin{aligned} W_2(12) &= (1 + P_{12}) f_1(2) + \int d1'' d1''' C(12; 1''') \\ &\quad \times C^{-1}(1'''; 1'') P_{11''} W_3(123) \\ &= (1 + P_{12} + P_{13}) f_2(23) + (1 + P_{13} + P_{23}) f_1(3) \\ &\quad + \int d1''' d1'' C(12; 1''') C^{-1}(1'''; 1'') P_{11''} \\ &\quad + \int d1''' d1'' C(123; 1''') C^{-1}(1'''; 1'') P_{11''} \end{aligned} \quad (\text{A8})$$

etc.

The correlation functions in Eq. (A8) may be expressed in terms of the distribution functions by using Eq. (2.23) and (2.26), with the result

$$\begin{aligned} &\int d1''' C(1, \dots, s; 1''') C^{-1}(1'''; 1'') \\ &= n^{s-1} \prod_{i=1}^s \phi(v_i) \left[h_s(\mathbf{r}_1, \dots, \mathbf{r}_s) \sum_{j=1}^s \right. \\ &\quad \times P_{ij} [\delta(1-1'') \phi^{-1}(v_1) - nc(\mathbf{r}_1, \mathbf{r}_{1''})] + nh_{s+1}(\mathbf{r}_1, \dots, \mathbf{r}_s, \mathbf{r}_{1''}) - n^2 \\ &\quad \left. \times \int d\mathbf{r}_{1''} h_{s+1}(\mathbf{r}_1, \dots, \mathbf{r}_s, \mathbf{r}_{1''}) c(\mathbf{r}_{1''}, \mathbf{r}_{1''}) \right] \end{aligned} \quad (\text{A9})$$

Substituting Eq. (A9) for $s=2, 3, \dots$ into Eq. (A8) and then inserting the virial expansion for the various distribution functions, as given in

Eqs. (A1), (A5), and (A6), the cluster expansion for the W_s is readily obtained. The details are quite lengthy, but straightforward and are not given here.

To lowest order in the density W_s is given by

$$W_s^{(0)}(1, \dots, s) = \sum_{i=1}^s P_{1i} \prod_{j=2}^s \phi(v_j) g_s^{(0)}(\mathbf{r}_1, \dots, \mathbf{r}_s) \quad (\text{A10})$$

with $g_s^{(0)}$ given by Eq. (A3). The first few higher-order terms are given here only for $s=2$,

$$\begin{aligned} W_2^{(1)}(1, 2) &= (1 + P_{12}) \phi(v_2) g_2^{(1)}(\mathbf{r}_1, \mathbf{r}_2) \\ &+ \phi(v_1) \phi(v_2) \int d\mathbf{l}'' [f_{11''} f_{1''2} + f_{12} f_{21''} f_{1''1}] P_{11''} \end{aligned} \quad (\text{A11a})$$

and

$$\begin{aligned} W_2^{(2)}(1, 2) &= (1 + P_{12}) \phi(v_2) g_2^{(2)}(\mathbf{r}_1, \mathbf{r}_2) \\ &+ \phi(v_1) \phi(v_2) \\ &\times \int d\mathbf{l}'' [f_{31''} f_{1''1} + f_{31''} f_{1''2} + f_{11''} f_{1''2} + O(f^3)] P_{11''} \end{aligned} \quad (\text{A11b})$$

APPENDIX B: HOMOGENEOUS AND MARKOVIAN FORM OF THE BGL OPERATOR

In this Appendix it is shown that in the limit $z \rightarrow 0^+$ and $\mathbf{k} \rightarrow 0$ the generalized BGL operator given in Eq. (3.13) reduces to the Markovian form used in the literature for a homogeneous gas and given in Eq. (3.17). The derivation follows closely that presented in Ref. 17 for the nonlinear BGL operator. The terms in Eq. (3.13) containing static correlations are essential to recover the usual form, Eq. (3.17).

Consider the action of $A_{\mathbf{k}}(\mathbf{v}_1, z)$ given by Eq. (3.13) on a function $A_{\mathbf{k}}(\mathbf{v}_1)$. The limit $\mathbf{k} \rightarrow 0$ of the operator may be taken from the onset, with the result

$$\begin{aligned} &A_0(\mathbf{v}_1, z) n\phi(v_1) A_{\mathbf{k}}(\mathbf{v}_1) \\ &= \frac{n^2}{m^2} \int d\mathbf{v}_2 \int \frac{d\mathbf{q}}{(2\pi)^3} \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} \frac{1}{z - i\mathbf{q} \cdot \mathbf{v}_{12} - \mathcal{V}_{-\mathbf{q}}(\mathbf{v}_1) - \mathcal{V}_{\mathbf{q}}(\mathbf{v}_2)} S_{\text{DH}}(q) \\ &\quad \times \phi(v_1) \phi(v_2) \left\{ \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} A_{\mathbf{k}}(\mathbf{v}_1) - \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_2} A_{\mathbf{k}}(\mathbf{v}_2) \right\}. \end{aligned} \quad (\text{B1})$$

The imaginary part of z can be set equal to zero in Eq. (B1); the limit $z \rightarrow 0^+$ is intended here and below. To proceed, it is convenient to introduce the Green's function of the Vlasov equation, defined as the solution of

$$\left[\frac{\partial}{\partial t} + i\mathbf{q} \cdot \mathbf{v}_1 - \mathcal{V}_q^-(\mathbf{v}_1) \right] U_{\mathbf{q}}(\mathbf{v}_1, t | \mathbf{v}_{1'}, 0) = \delta(t) \delta(\mathbf{v}_1 - \mathbf{v}_{1'}) \quad (\text{B2a})$$

and given by

$$U_{\mathbf{q}}(\mathbf{v}_1, t | \mathbf{v}_{1'}, 0) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2\pi i} e^{zt} U_{\mathbf{q}z}(\mathbf{v}_1 | \mathbf{v}_{1'}) \quad (\text{B2b})$$

for $\text{Re}z > 0$, with

$$U_{\mathbf{q}z}(\mathbf{v}_1 | \mathbf{v}_{1'}) = \frac{1}{z + i\mathbf{q} \cdot \mathbf{v}_1} \left\{ \delta(\mathbf{v}_1 - \mathbf{v}_{1'}) + \left[\boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} n\phi(v_1) \right] \frac{1}{D(\mathbf{q}, z)} \frac{1}{z + i\mathbf{q} \cdot \mathbf{v}_{1'}} \right\} \quad (\text{B2c})$$

where $D(\mathbf{q}, z)$ is the plasma dispersion function, defined in Eq. (3.16b). The contour of integration in Eq. (B.2b) is a vertical line in the complex z plane, with $\gamma > 0$, located to the right of all singularities of the integrand. Equation (B1) can be written

$$\begin{aligned} & A_0(\mathbf{v}_1, z) n\phi(v_1) A_{\mathbf{k}}(\mathbf{v}_1) \\ &= \frac{n^2}{m^2} \int \frac{d\mathbf{q}}{(2\pi)^3} \int_{\gamma_1-i\infty}^{\gamma_1+i\infty} \frac{dz_1}{2\pi i} \int_{\gamma_2-i\infty}^{\gamma_2+i\infty} \frac{dz_2}{2\pi i} \frac{1}{z - (z_1 + z_2)} \\ & \quad \times \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} \int d\mathbf{v}_2 d\mathbf{v}_{1'} d\mathbf{v}_{2'} S_{\text{DH}}(q) \\ & \quad \times U_{-\mathbf{q}z_1}(\mathbf{v}_1 | \mathbf{v}_{1'}) U_{\mathbf{q}z_2}(\mathbf{v}_2 | \mathbf{v}_{2'}) \phi(v_{1'}) \phi(v_{2'}) \\ & \quad \times \left[\boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_{1'}} A_{\mathbf{k}}(\mathbf{v}_{1'}) + \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_{2'}} A_{\mathbf{k}}(\mathbf{v}_{2'}) \right] \end{aligned} \quad (\text{B3})$$

with $\text{Re}(z - (z_1 + z_2)) > 0$. The \mathbf{v}_2 integration may be performed, using

$$\int d\mathbf{v}_2 U_{\mathbf{q}z_2}(\mathbf{v}_2 | \mathbf{v}_{2'}) = \frac{1}{z_2 + i\mathbf{q} \cdot \mathbf{v}_{2'}} \frac{1}{D(\mathbf{q}z_2)} \quad (\text{B4})$$

Also it is convenient to define the following functions:

$$\mathcal{A}(\mathbf{q}, z) = \int d\mathbf{v} \frac{1}{z + i\mathbf{q} \cdot \mathbf{v}} n\phi(v) \quad (\text{B5})$$

and

$$\hat{D}(\mathbf{q}, z) = 1 - \int d\mathbf{v} \frac{1}{z + i\mathbf{q} \cdot \mathbf{v}} n\phi(v) \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}} A_{\mathbf{k}}(\mathbf{v}) \quad (\text{B6})$$

for $\text{Re}z > 0$. We assume that the unknown function $A_{\mathbf{k}}(\mathbf{v}_1)$ is such that $\hat{D}(\mathbf{q}, z)$ has the same analytic properties as $D(\mathbf{q}, z)$, i.e., $\hat{D}(\mathbf{q}, z) \neq 0$, for $\text{Re}z > 0$ and all $\mathbf{q} \neq 0$. Making use of the above definitions and of the explicit form of the Vlasov propagators, Eq. (B3) may be written as follows:

$$\begin{aligned} & A_0(\mathbf{v}_1, z) n\phi(v_1) A_{\mathbf{k}}(\mathbf{v}_1) \\ &= \frac{1}{m^2} \int \frac{d\mathbf{q}}{(2\pi)^3} S_{\text{DH}}(q) \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{dz_1}{2\pi i} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \\ & \times \frac{dz_2}{2\pi i} \frac{1}{z - (z_1 + z_2)} \frac{1}{z_1 - i\mathbf{q} \cdot \mathbf{v}_1} \frac{1}{D(\mathbf{q}, z_2)} \\ & \times \left\{ \mathcal{A}(\mathbf{q}, z_2) n\phi(v_1) \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} A_{\mathbf{k}}(\mathbf{v}_1) + n\phi(v_1) [1 - \hat{D}(\mathbf{q}, z_2)] \right. \\ & + \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} n\phi(v_1) \frac{1}{D(-\mathbf{q}, z_1)} [\mathcal{A}(\mathbf{q}, z_2)(1 - \hat{D}(-\mathbf{q}, z_1)) \\ & \left. + \mathcal{A}(-\mathbf{q}, z_1)(1 - \hat{D}(\mathbf{q}, z_2))] \right\} \quad (\text{B7}) \end{aligned}$$

We now define the analytic continuation of the functions \mathcal{A} , D , and \hat{D} to $\text{Re}z < 0$. This will give us more freedom in dealing with the contours of integration. Such analytic continuations are given for $\text{Re}z < 0$ by

$$\mathcal{A}_-(\mathbf{q}, z) = \int d\mathbf{v} \frac{1}{z + i\mathbf{q} \cdot \mathbf{v}} n\phi(v) - 2\pi i \int d\mathbf{v} \delta(z + i\mathbf{q} \cdot \mathbf{v}) n\phi(v) \quad (\text{B8a})$$

$$\begin{aligned} D_-(\mathbf{q}, z) &= 1 - \int d\mathbf{v} \frac{1}{z + i\mathbf{q} \cdot \mathbf{v}} \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}} n\phi(v) \\ & - 2\pi i \int d\mathbf{v} \delta(z + i\mathbf{q} \cdot \mathbf{v}) \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}} n\phi(v) \quad (\text{B8b}) \end{aligned}$$

$$\begin{aligned} \hat{D}_-(\mathbf{q}, z) &= 1 - \int d\mathbf{v} \frac{1}{z + i\mathbf{q} \cdot \mathbf{v}} n\phi(v) \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}} A_{\mathbf{k}}(\mathbf{v}) \\ & - 2\pi i \int d\mathbf{v} \delta(z + i\mathbf{q} \cdot \mathbf{v}) n\phi(v) \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}} A_{\mathbf{k}}(\mathbf{v}) \quad (\text{B8c}) \end{aligned}$$

By letting $z = i\omega \pm \varepsilon$ in Eqs. (B8) and using that the singular function $\lim_{\varepsilon \rightarrow 0^+} (\varepsilon + ix)^{-1}$ can be written as the sum of a δ function and a Cauchy principal value, it can be easily verified that the functions of Eqs. (B8) and those defined in Eqs. (316b) and (B5), (B6) are the same in the limit $\varepsilon \rightarrow 0^+$.

The z_1 integration may be performed in the first two terms in curly brackets in Eq. (B7) by closing the contour to the left (where the only singularity is the pole $z_1 = z - z_2$). Similarly in the third term in the curly brackets the z_2 integration is performed by closing the contour to the left. The result is

$$\begin{aligned}
 & A_0(\mathbf{v}_1, z) n\phi(v_1) A_{\mathbf{k}}(\mathbf{v}_1) \\
 &= \frac{1}{m^2} \int \frac{d\mathbf{q}}{(2\pi)^3} S_{\text{DH}}(q) \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} \left\{ \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} \frac{dz_2}{2\pi i} \frac{1}{z - z_2 - i\mathbf{q} \cdot \mathbf{v}_1} \frac{1}{D(\mathbf{q}, z_2)} \right. \\
 & \quad \times \left[\mathcal{A}(\mathbf{q}, z_2) n\phi(v_1) \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} A_{\mathbf{k}}(\mathbf{v}_1) + n\phi(v_1)(1 - \hat{D}(\mathbf{q}, z_2)) \right] \\
 & \quad + \left(\boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} n\phi(v_1) \right) \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \frac{dz_1}{2\pi i} \frac{1}{z_1 - i\mathbf{q} \cdot \mathbf{v}_1} \\
 & \quad \times \frac{1}{D(\mathbf{q}, z - z_1) D(-\mathbf{q}, z_1)} \\
 & \quad \left. \times [\mathcal{A}(\mathbf{q}, z - z_1)(1 - \hat{D}(-\mathbf{q}, z_1)) + \mathcal{A}(-\mathbf{q}, z_1)(1 - \hat{D}(\mathbf{q}, z - z_1))] \right\} \tag{B9}
 \end{aligned}$$

The residual z_2 integration in the first term in Eq. (B9) can now be performed by closing the contour to the right. To perform the z_1 integration in the second term, it is convenient to push the contour of integration against the real axis, by letting $z_1 = i\omega + \eta$, where $\eta \rightarrow 0^+$ is intended. The result is

$$\begin{aligned}
 & A_0(\mathbf{v}_1, z) n\phi(v_1) A_{\mathbf{k}}(\mathbf{v}_1) \\
 &= \frac{1}{m^2} \int \frac{d\mathbf{q}}{(2\pi)^3} S_{\text{DH}}(q) \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} \left\{ \frac{\mathcal{A}(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)}{\mathcal{D}(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)} n\phi(v_1) \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} A_{\mathbf{k}}(\mathbf{v}_1) \right. \\
 & \quad + \frac{1 - \hat{D}(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)}{D(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)} n\phi(v_1) \\
 & \quad + \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{1}{\eta + i\omega - i\mathbf{q} \cdot \mathbf{v}_1} \left(\boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} n\phi(v_1) \right) \\
 & \quad \left. \times \frac{1}{|D(\mathbf{q}, -i\omega)|^2} 2\text{Re}[\mathcal{A}(\mathbf{q}, i\omega)(1 - \hat{D}(-\mathbf{q}, i\omega))] \right\} \tag{B10}
 \end{aligned}$$

where use has been made of the property $F(\mathbf{q}, -i\omega) = [F(-\mathbf{q}, i\omega)]^*$, where the star indicates complex conjugation and F is any of the functions \mathcal{A} , D , and \hat{D} . Changing \mathbf{q} into $-\mathbf{q}$ and carrying out the ω integral, Eq. (B10) becomes

$$\begin{aligned}
& A_0(\mathbf{v}_1, 0) n\phi(v_1) A_{\mathbf{k}}(\mathbf{v}_1) \\
&= \frac{1}{m^2} \int \frac{d\mathbf{q}}{(2\pi)^3} S_{\text{DH}}(q) \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} \frac{1}{|D(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)|^2} \\
&\quad \times \left\{ \text{Re}[D(-\mathbf{q}, i\mathbf{q} \cdot \mathbf{v}_1) \mathcal{A}(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)] n\phi(v_1) \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} A_{\mathbf{k}}(\mathbf{v}_1) \right. \\
&\quad + i\text{Im}[D(-\mathbf{q}, i\mathbf{q} \cdot \mathbf{v}_1)(1 - \hat{D}(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1))] n\phi(v_1) \\
&\quad \left. + \text{Re}[\mathcal{A}(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)(1 - \hat{D}(-\mathbf{q}, i\mathbf{q} \cdot \mathbf{v}_1))] \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} n\phi(v_1) \right\} \quad (\text{B11})
\end{aligned}$$

The real and imaginary part of the functions appearing in Eq. (B11) can be evaluated by using Eqs. (B8) for $z = \varepsilon \pm i\mathbf{q} \cdot \mathbf{v}_1$ and letting $\varepsilon \rightarrow 0^+$. Equation (B11) becomes then

$$\begin{aligned}
& A_0(\mathbf{v}_1, 0) n\phi(v_1) A_{\mathbf{k}}(\mathbf{v}_1) \\
&= \frac{\pi}{m^2} \int \frac{d\mathbf{q}}{(2\pi)^3} \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{v}_1} \frac{V_{\mathbf{q}}^2}{|D(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1)|^2} S_{\text{DH}}(q) (1 + n\beta V_{\mathbf{q}}) \\
&\quad \times \int d\mathbf{v}_2 \delta(\mathbf{q} \cdot \mathbf{v}_{12}) n\phi(v_1) n\phi(v_2) \\
&\quad \times \mathbf{q} \left(\frac{\partial}{\partial \mathbf{v}_1} - \frac{\partial}{\partial \mathbf{v}_2} \right) (1 + P_{12}) A_{\mathbf{k}}(\mathbf{v}_1) \quad (\text{B12a})
\end{aligned}$$

with

$$\begin{aligned}
D(\mathbf{q}, -i\mathbf{q} \cdot \mathbf{v}_1) &= 1 + \frac{1}{m} V_{\mathbf{q}} P \int d\mathbf{v}_2 \frac{1}{\mathbf{q} \cdot \mathbf{v}_{12}} \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{v}_2} n\phi(v_2) \\
&\quad - \frac{i\pi}{m} V_{\mathbf{q}} \int d\mathbf{v}_2 \delta(\mathbf{q} \cdot \mathbf{v}_{12}) \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{v}_2} n\phi(v_2) \quad (\text{B12b})
\end{aligned}$$

where P denotes the Cauchy principal part of the integral. Finally, using $1 + n\beta V_{\mathbf{q}} = [S_{\text{DH}}(q)]^{-1}$, Eq. (B12a) reduces to Eq. (3.16). We remark that the same Eq. (B12) would also have been obtained if we had neglected the functions $\bar{\theta}_{\mathbf{q}}$ in Eq. (3.13). Again, this is due to the fact that the Markovian limit of the BGL operator conserves kinetic energy.

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